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Best Local Approximations in $L^p(\mu)$

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I. INTRODUCTION

The purpose of this note is to generalize a recent result of Macías and Zó in [4] concerning weighted best local L^p approximation. Results of this type have their origin in the work of Freud [3] and Maehly and Witzgall [5].

We consider a positive Borel measure, μ, on the unit ball, B, in R^n, with $\mu(B) = 1$. This measure is required to be nondegenerate in the sense that it is not supported in the zero set of a nontrivial polynomial and that $\mu(\varepsilon B) > 0$ for all ε in $(0, 1]$ where $\varepsilon E := \{y \in R^n: y = \varepsilon x, x \in E\}$. The dilates of μ are the measures, μ_ε, $0 < \varepsilon \leqslant 1$, given at the Borel set $E \subset B$ by $\mu_\varepsilon(E) = \mu(\varepsilon E)/\mu(\varepsilon B)$.

As usual, $L^p(\mu)$, $1 < p < \infty$, is the class of all measurable functions, f, on B such that $\|f\|_{p,\mu} := [\int_B |f(x)|^p \, d\mu]^{1/p} < \infty$. Given $f \in L^p(\mu)$, we denote by $P_{m,\mu} f$ the unique element of π_m, the class of real polynomials of degree at most m, satisfying $\|f - p_{m,\mu} f\|_{p,\mu} = \inf_{P \in \pi_m} \|f - P\|_{p,\mu}$. Restricting attention to a special class of measures $d\mu = w(|x|)\,dx$ (see Example 2.4(1), below), Macías and Zó studied the limiting behaviour, as $\varepsilon \to 0+$, of

$$(E_\varepsilon f)(t) := \varepsilon^{-m-1}[f(\varepsilon t) - (P_{m,\mu_\varepsilon} f(\varepsilon \cdot))(\varepsilon t)], \qquad |t| \leqslant 1,$$

when f belongs to a certain subspace of $L^p(\mu)$. We observe that $(P_{m,\mu_\varepsilon} f(\varepsilon \cdot))(t) = P_m(\varepsilon t)$, where P_m is the best approximation out of π_m to f in L^p with respect to the measure $\mu(\cdot)/\mu(\varepsilon B)$ on B. Our main result, Theorem 2.3, shows $E_\varepsilon f$ behaves the same way for a much larger class of μ. The key to proving this is in finding a substitute for the weight $\tilde{w}$ associated with the given weight w in [4], as well as for the important Lemma 1 concerning it. This is found in the measure ν associated with the given measure μ, in Lemmas 2.1 and 2.2, below.

We will use the customary notation $C(K)$ for the space of continuous,

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real-valued functions on the compact set K in R^n; we denote the uniform norm by $\| \ \|_\infty$.

II. Results and Examples

The proof of the following result is essentially given in [2, p. 243].

Lemma 2.1. *Let μ be a positive Borel measure on B, $\mu(B)=1$. Then, a necessary and sufficient condition for there to be another such measure ν so that*

$$\lim_{\varepsilon \to 0+} \int_B g(x)\, d\mu_\varepsilon = \int_B g(x)\, d\nu, \qquad g \in C(B) \tag{2.1}$$

is the existence of the limits

$$\lim_{\varepsilon \to 0+} \int_B x_j^k\, d\mu_\varepsilon, \qquad j=1, ..., n; \quad k=0, 1, ... \tag{2.2}$$

Here $x=(x_1, ..., x_n) \in R^n$.

Lemma 2.2. *Let μ and ν be positive, nondegenerate Borel measures on B, $\mu(B)=\nu(B)=1$, satisfying (2.1). Then the norms $\| \ \|_{p,\mu_\varepsilon}$ and $\| \ \|_{p,\nu}$ are equivalent on π_m, independently of ε in $(0, 1]$, for each fixed $m \in Z_+$.*

Proof. Letting the linear functionals F_ε and F be as in Lemma 2.1 we show the ratio

$$\|P\|_{p,\mu_\varepsilon}^p / \|P\|_{p,\nu}^p = F_\varepsilon(|P|^p)/F(|P|^p), \qquad P \not\equiv 0$$

is bounded above independently of ε in $(0, 1]$ and $P \in \pi_m$. The proof for the reciprocal ratio is the same.

If the ratio were not bounded, then there would exist sequences $\varepsilon_k \downarrow 0$ and $P_k \in \pi_m$, $\|P_k\|_\infty = 1$, such that $F_{\varepsilon_k}(|P_k|^p)/F(|P_k|^p) > k$, $k=1, 2, ...$ However, the compactness of the unit sphere in π_m with respect to $\| \ \|_\infty$ allows us to further assume there is a $P \in \pi_m$, $\|P\|_\infty = 1$, with $\lim_{k\to\infty} \|P - P_k\|_\infty = 0$ and so $\lim_{k\to\infty} \| |P|^p - |P_k|^p \|_\infty = 0$. Since the F_ε are uniformly bounded, this would mean $\lim_{k\to\infty} F_{\varepsilon_k}(|P_k|^p)/F(|P_k|^p) = \lim_{k\to\infty} F_{\varepsilon_k}(|P|^p)/F(|P_k|) = 1$, a contradiction.

The special subspace of $L^p(\mu)$ referred to in the first section is

$$t_{m,\mu}^p := \{ f \in L^p(\mu) : \|f(\varepsilon\cdot) - T_m(\varepsilon\cdot)\|_{p,\mu_\varepsilon} = o(\varepsilon^m) \text{ for some } T_m \in \pi_m \}.$$

See [1, 3]. Lemma 2.2, above, implies the polynomial T_m corresponding to $f \in t^p_{m,\mu}$ is unique, if μ satisfies (2.1).

We now have all the ingredients to prove

THEOREM 2.3. *Let μ be a positive Borel measure on B, $\mu(B)=1$, for which $\lim_{\varepsilon \to 0+} \int_B x_j^k \, d\mu_\varepsilon$ exists, $j=1, ..., n$; $k=0, 1, ...$ Let ν be the measure guaranteed by Lemma 2.1 to satisfy*

$$\lim_{\varepsilon \to 0+} \int_B g(x) \, d\mu_\varepsilon = \int_B g(x) \, d\nu, \qquad g \in C(B). \tag{2.3}$$

Suppose that both μ and ν are nondegenerate. For $f \in t^p_{m+1,\mu}$, set $\phi_{m+1} = T_{m+1} - T_m$. Then,

$$E_\varepsilon f = \varepsilon^{-m-1}[f(\varepsilon t) - (P_{m,\mu_\varepsilon} f(\varepsilon \cdot))(t)]$$

satisfies

(i) $\lim_{\varepsilon \to 0+} \|E_\varepsilon f - (\phi_{m+1} - P)\|_{p,\mu_\varepsilon} = 0$

(ii) $\lim_{\varepsilon \to 0+} \|E_\varepsilon f\|_{p,\mu_\varepsilon} = \|(\phi_{m+1} - P)\|_{p,\nu}$,

where $P = P_{m,\nu} \phi_{m+1}$.

Proof. To begin, we observe that, by (2.3), (i) implies (ii), and so it is enough to prove (i).

Since $f \in t^p_{m+1,\mu}$,

$$f(\varepsilon t) = T_{m+1}(\varepsilon t) + \varepsilon^{m+1} R_\varepsilon(t), \qquad |t| < 1,$$

where

$$\lim_{\varepsilon \to 0+} \|R_\varepsilon\|_{p,\mu_\varepsilon} = 0. \tag{2.4}$$

Set

$$q_\varepsilon(t) := \varepsilon^{-m-1}[(P_{m,\mu_\varepsilon} f(\varepsilon \cdot))(t) - T_m(\varepsilon t)].$$

Then, $q_\varepsilon = P_{m,\mu_\varepsilon}(h_\varepsilon)$, $h_\varepsilon = \phi_{m+1} + R_\varepsilon$, with $\|q_\varepsilon\|_{p,\nu}$ uniformly bounded, in view of (2.4) and Lemma 2.2. Also, assertion (i) can be written

$$\lim_{\varepsilon \to 0+} \|P - q_\varepsilon\|_{p,\mu_\varepsilon} = 0. \tag{2.5}$$

Now, if (2.5) were not true, the compactness of the unit sphere in π_m and Lemma 2.2 would ensure the existence of a sequence $\varepsilon_k \downarrow 0$ and $q \in \pi_m$, $q \neq P$, such that

$$\lim_{k \to \infty} \|q - q_{\varepsilon_k}\|_{p,\mu_{\varepsilon_k}} = 0,$$

and hence

$$\lim_{k \to \infty} \|h_{\varepsilon_k} - P\|_{p,\mu_{\varepsilon_k}} = \|\phi_{m+1} - P\|_{p,\nu} < \|\phi_{m+1} - q\|_{p,\nu}$$

$$= \lim_{k \to \infty} \|h_{\varepsilon_k} - q_{\varepsilon_k}\|_{p,\mu_{\varepsilon_k}}.$$

This is incompatible, for sufficiently large k, with the minimality of q_{ε_k}.

EXAMPLES 2.4. (1) The measures μ considered in [4] are of the form $d\mu(x) = w(|x|)\,dx$, where

$$\lim_{\varepsilon \to 0+} \omega_n^{-1} \varepsilon^{-(\beta+n)} \int_{|x| \leqslant \varepsilon} w(|x|)\,dx = \lim_{\varepsilon \to 0+} \varepsilon^{-(\beta+n)} \int_0^{\varepsilon} r^{n-1} w(r)\,dr := A > 0,$$

exists for some $\beta > -n$, and ω_n denotes the surface area of B. Thus, essentially, $w(|x|)$ behaves like $|x|^{\beta}$. Wle claim that such μ satisfy (2.1) and, further, the associated measure ν is given by $d\nu(x) = \tilde{w}(|x|)\,dx$, $\tilde{w}(|x|) = \omega_n^{-1}(\beta+n)\,|x|^{\beta}$. To prove this it suffices to show that

$$\lim_{\varepsilon \to 0+} \int_B x_j^k \, d\mu_\varepsilon(x) = \lim_{\varepsilon \to 0+} \frac{\varepsilon^n \int_{|x| \leqslant 1} x_j^k w(\varepsilon |x|)\,dx}{\int_{|x| \leqslant \varepsilon} w(|x|)\,dx}$$

$$= \lim_{\varepsilon \to 0+} \frac{c_j \varepsilon^{-k} \int_0^{\varepsilon} r^{k+n-1} w(r)\,dr}{\int_0^{\varepsilon} r^{n-1} w(r)\,dr}$$

$$= \frac{\beta+n}{\beta+n+k}\, c_j, \qquad j = 1, ..., n,$$

where c_j is independent of ε; indeed, it is enough to show that

$$\lim_{\varepsilon \to 0+} \varepsilon^{-(\beta+n+k)} \int_0^{\varepsilon} r^{k+n-1} w(r)\,dr = \frac{\beta+n}{\beta+n+k} A, \qquad k = 0, 1, \ldots.$$

But, integrating by parts, we find

$$\varepsilon^{-(\beta+n+k)} \int_0^{\varepsilon} r^{k+n-1} w(r)\,dr = \varepsilon^{-(\beta+n)} \int_0^{\varepsilon} r^{n-1} w(r)\,dr$$

$$- k\varepsilon^{-(\beta+n+k)} \int_0^{\varepsilon} r^{k-1} W(r)\,dr,$$

with $W(r) := \int_0^r s^{n-1} w(s)\,ds$. Finally, l'Hôpital's rule yields

$$\lim_{\varepsilon \to 0+} e^{-(\beta+n+k)} \int_0^{\varepsilon} r^{k-1} W(r)\,dr = \frac{A}{\beta+n+k}.$$

(2) A nonnegative function, ϕ, defined on (0, 1] is said to be slowly varying near 0, in the sense of Hardy and Rogosinski, if to each $\delta > 0$, there corresponds $t_0 = t_0(\delta)$ in (0, 1] such that $t^{\delta}\phi(t)$ is nondecreasing and $t^{-\delta}\phi(t)$ is nonincreasing on $(0, t_0)$. Given μ with $d\mu(x) = c\,|x|^{\beta}\,\phi(|x|)\,dx$, where $\beta > -n$, and c is a normalizing constant, standard arguments show ν satisfies $d\nu(x) = \omega_n^{-1}(\beta + n)\,|x|^{\beta}\,dx$. See [6, p. 186].

(3) Measures μ of the form $d\mu(x) = c(\prod_{i=1}^{n} |x_i|^{\beta_i})\,dx$, $x = (x_1, ..., x_n)$, $\beta_i > -1$, give rise to $\nu = \mu$, since

$$\mu(\varepsilon E) = c \int_{\varepsilon E} \left(\prod_{i=1}^{n} |x_i|^{\beta_i} \right) dx = \varepsilon^{n + \sum \beta_i} c \int_E \left(\prod_{i=1}^{n} |y_i|^{\beta_i} \right) dy,$$

from which it follows that $\mu_{\varepsilon}(E) = \mu(E)$.

(4) When $d\mu(x) = c(\prod_{i=1}^{n} |x_i|^{\beta_i}\,\phi_i(x_i))\,dx$, $\beta_i > -1$, ϕ_i slowly varying, then ν satisfies $d\nu(x) = k\prod_{i=1}^{n} |x_i|^{\beta_i}\,dx$.

(5) Measures μ that have either the form $d\mu(x) = c\,|x|^{-n}\,[\log e/|x|]^{\beta}$, $\beta < -n$, or the form $d\mu(x) = ce^{-1/|x|}\,dx$, give rise to degenerate ν, in fact, to the Dirac delta measure and the singular normalized surface measure on the unit sphere, respectively.

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